JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 17, No. 2, March–April 1994

# Some Basic Considerations on Angles Describing Airplane Flight Maneuvers

Osamu Kato*
*Nagoya University, Nagoya 464, Japan*

**Several angles to describe the flight maneuvers of an airplane are introduced. First, relative Euler angles, which show one flight attitude relative to another flight attitude, are defined. Next, rolling angle, pitching angle, and yawing angle, which are other than Euler angles, are defined. In addition, nose moving angle is defined as a quantity that shows the length of the attitude path along which the flight attitude changes. Comparative studies between similar angles in the small-disturbance theory are made in large flight maneuvers. A formula which relates the rolling angle to the variation of roll angle is obtained by means of the attitude projection method and Stokes theorem. It is shown that although the differences between the relative roll angle, rolling angle, and variation of roll angle are sufficiently small in small-disturbance flight mechanics; those differences become considerably larger in large flight maneuvers. Numerical examples are presented to illustrate these results.**

## Nomenclature

$e_x$, $e_y$, $e_z$ = unit vectors of body axes
$p$, $q$, $r$ = angular velocity components: rolling rate, pitching rate, and yawing rate, respectively
$\int p\,dt$, $\int q\,dt$, $\int r\,dt$ = rolling angle, pitching angle, and yawing angle, respectively
$w_{\psi\text{-}\theta}$ = velocity of $\psi$-$\theta$ trajectory
$\psi$, $\theta$, $\phi$ ; $\psi_i$, $\theta_i$, $\phi_i$ = yaw Euler angle (heading angle), pitch Euler angle (pitch angle), and roll Euler angle (bank angle), respectively.
$\psi_r$, $\theta_r$, $\phi_r$ = relative yaw Euler angle, relative pitch Euler angle, and relative roll Euler angle, respectively

## Introduction

THIS paper discusses several kinds of angles used to describe airplane flight maneuvers. Attitude angles, which represent the instantaneous flight attitude of an airplane, of course, are included. The most common attitude angles are Euler angles. The three Euler angles $\psi$, $\theta$, and $\phi$ used in flight mechanics are called "yaw angle (heading angle)," "pitch angle," and "roll angle (bank angle)," respectively.

On the other hand, the terms "yawing," "pitching," and "rolling" mean the angular motions about the airplane-fixed $Z$, $Y$, and $X$ axes, respectively. These angular motions are usually described by the angular velocity components $r$, $q$, and $p$ with respect to the $e_z$, $e_y$, and $e_x$ body-fixed unit vectors, respectively. Then, by how many degrees of angle does a maneuvering airplane rotate about each axis from one moment to another? These rotation angles are not equal to the variations of Euler angles during that time. The Euler angles are kinds of rotation angles. However, those define a special rotation sequence and not, in general, the rotation angles of the actual rotation sequence at that time.

For example, suppose that the bank angle of an airplane at time $t_1$ was 30 deg, and the bank angle at time $t_2$ became 45 deg. Then, can we say that the airplane rotated 15 deg to the right about the $X$ axis during this time interval $[t_1, t_2]$? The answer is almost "yes" assuming the linearized, small-disturbance flight mechanics. In the large-maneuver flight mechanics, however, it is "no" in general. In this paper, these rotation angles about the body-fixed $X$, $Y$, and $Z$ axes during the actual maneuver will be called "rolling angle," "pitching angle," and "yawing angle," respectively. Mathematically, these angles are the time integrals of the angular velocity component $p$, $q$, and $r$, respectively. Note that these are not the same as "roll Euler angle," "pitch Euler angle," and "yaw Euler angle" stated initially. The names of these two groups may be confusing, so the term "Euler" is added to all of the latter angles to avoid such confusion hereafter.

For another problem, when we consider two flight attitudes that are shown by Euler angles $(\psi_1, \theta_1, \phi_1)$ and $(\psi_2, \theta_2, \phi_2)$, how do we compare the two attitudes? Are those attitudes near or far? How is the relative attitude viewed from one attitude to another? For the first question, as is easily seen, the square root of $(\psi_2 - \psi_1)^2 + (\theta_2 - \theta_1)^2 + (\phi_2 - \phi_1)^2$ is not suitable for measuring the distance between the two flight attitudes. To answer the second question, something like what are called "relative Euler angles" are required.

The motives for this study are the questions just stated. These questions may occur to anyone who tries to treat large flight maneuvers precisely. But the problem with angles is not as easy to handle as problems with other physical quantities. In flight mechanics, the methods of considering angles on the surface of a sphere have been introduced recently by Kalviste[1] and Kato,[2] independently. This study uses and develops the latter method and makes clear the problems regarding some angles.

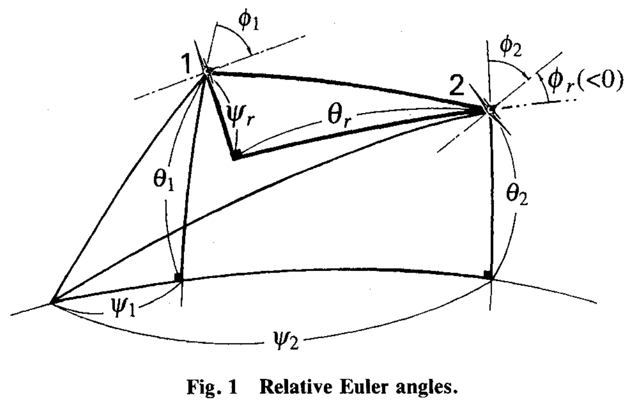


**Fig. 1 Relative Euler angles.**

Received May 2, 1992; revision received Nov. 24, 1992; accepted for publication March 27, 1993. 

*Assistant Professor, Department of Aeronautical Engineering, Chikusa-ku. Member AIAA.

## Relative Euler Angles

Consider the Euler angles of flight attitude 1 and flight attitude 2 of a maneuvering airplane, $(\psi_1, \theta_1, \phi_1)$ and $(\psi_2, \theta_2, \phi_2)$. The Euler angles are inconvenient when considering the compositions and the inverses of rotations. For example, it is not possible to show the Euler rotation from flight attitude 1 to flight attitude 2 by the variations of Euler angles, $(\psi_2 - \psi_1, \theta_2 - \theta_1, \phi_2 - \phi_1)$. Therefore, in this paper the relative Euler angles are defined as follows.

Let the Euler angles of the rotation from flight attitude 1 to flight attitude 2 be $(\psi_r, \theta_r, \phi_r)$. The angles $\psi_r$, $\theta_r$, and $\phi_r$ are taken as if flight attitude 1 were fixed as the usual reference attitude. If $(\psi_1, \theta_1, \phi_1) = (0, 0, 0)$, the Euler angles $(\psi_r, \theta_r, \phi_r)$ agree with the normal Euler angles. Using the attitude projection method,[2] the graphical relationships between these angles are illustrated in Fig. 1. Note that this kind of figure must originally be drawn on the surface of a sphere (the $\psi$-$\theta$ surface). The points on the $\psi$-$\theta$ surface can be regarded as the projections of an airplane's nose on the celestial sphere as viewed from the cockpit of the airplane.

Also, let $R(\psi, \theta, \phi)$ be the Euler rotation represented by an orthogonal matrix,[3] i.e.,

$$R(\psi, \theta, \phi) = \begin{bmatrix} \cos\psi\cos\theta & \sin\psi\cos\theta & -\sin\theta \\ \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \cos\theta\sin\phi \\ \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi & \cos\theta\cos\phi \end{bmatrix} \tag{1}$$

Flight attitude 1 $(\psi_1, \theta_1, \phi_1)$ and flight attitude 2 $(\psi_2, \theta_2, \phi_2)$ are obtained through the Euler rotations $R(\psi_1, \theta_1, \phi_1)$ and $R(\psi_2, \theta_2, \phi_2)$, respectively, from the reference attitude $(\psi, \theta, \phi) = (0, 0, 0)$. These rotations and the relative rotation $R(\psi_r, \theta_r, \phi_r)$ can be related by the operation of orthogonal matrices:

$$R(\psi_r, \theta_r, \phi_r)\, R(\psi_1, \theta_1, \phi_1) = R(\psi_2, \theta_2, \phi_2) \tag{2}$$

Multiplying both sides of Eq. (2) by $R(\psi_1, \theta_1, \phi_1)^{-1}$ from the right

$$R(\psi_r, \theta_r, \phi_r) = R(\psi_2, \theta_2, \phi_2) R(\psi_1, \theta_1, \phi_1)^{-1} \tag{3}$$

where $R(\psi_1, \theta_1, \phi_1)^{-1}$ is the inverse of $R(\psi_1, \theta_1, \phi_1)$. Therefore, if $(\psi_1, \theta_1, \phi_1)$ and $(\psi_2, \theta_2, \phi_2)$ are given, we can obtain the "relative Euler angles" $(\psi_r, \theta_r, \phi_r)$ as follows.

By expanding Eq. (3) and comparing appropriate elements of the matrices on both sides, we have

$$\cos\psi_r \cos\theta_r = \cos\psi_2 \cos\psi_1 \cos\theta_2 \cos\theta_1 + \sin\psi_2 \cos\theta_2 \sin\psi_1 \cos\theta_1 + \sin\theta_2 \sin\theta_1 \tag{4}$$

$$\begin{aligned}\sin\psi_r \cos\theta_r = {} & \cos\psi_2 \cos\theta_2 (\cos\psi_1 \sin\theta_1 \sin\phi_1 - \sin\psi_1 \cos\phi_1) + \sin\psi_2 \cos\theta_2 (\sin\psi_1 \sin\theta_1 \sin\phi_1 \\ & + \cos\psi_1 \cos\phi_1) - \sin\theta_2 \cos\theta_1 \sin\phi_1 \end{aligned} \tag{5}$$

$$\begin{aligned}-\sin\theta_r = {} & \cos\psi_2 \cos\theta_2 (\cos\psi_1 \sin\theta_1 \cos\phi_1 + \sin\psi_1 \sin\phi_1) + \sin\psi_2 \cos\theta_2 (\sin\psi_1 \sin\theta_1 \cos\phi_1 \\ & - \cos\psi_1 \sin\phi_1) - \sin\theta_2 \cos\theta_1 \cos\phi_1 \end{aligned} \tag{6}$$

$$\begin{aligned}\cos\theta_r \sin\phi_r = {} & (\cos\psi_2 \sin\theta_2 \sin\phi_2 - \sin\psi_2 \cos\phi_2)(\cos\psi_1 \sin\theta_1 \cos\phi_1 + \sin\psi_1 \sin\phi_1) \\ & + (\sin\psi_2 \sin\theta_2 \sin\phi_2 + \cos\psi_2 \cos\phi_2)(\sin\psi_1 \sin\theta_1 \cos\phi_1 \\ & - \cos\psi_1 \sin\phi_1) + \cos\theta_2 \sin\phi_2 \cos\theta_1 \cos\phi_1 \end{aligned} \tag{7}$$

$$\begin{aligned}\cos\theta_r \cos\phi_r = {} & (\cos\psi_2 \sin\theta_2 \cos\phi_2 + \sin\psi_2 \sin\phi_2)(\cos\psi_1 \sin\theta_1 \cos\phi_1 + \sin\psi_1 \sin\phi_1) \\ & + (\sin\psi_2 \sin\theta_2 \cos\phi_2 - \cos\psi_2 \sin\phi_2)(\sin\psi_1 \sin\theta_1 \cos\phi_1 \\ & - \cos\psi_1 \sin\phi_1) + \cos\theta_2 \cos\phi_2 \cos\theta_1 \cos\phi_1 \end{aligned} \tag{8}$$

From Eqs. (4) and (5), $\psi_r$ can be calculated. Also, $\theta_r$ can be calculated from Eq. (6), and $\phi_r$ from Eqs. (7) and (8), respectively.

**Example 1**

Attitude 1 is $(\psi_1, \theta_1, \phi_1) = (0, 30, 180 \text{ deg})$ and attitude 2 is $(\psi_2, \theta_2, \phi_2) = (0, 0, 0 \text{ deg})$. By calculating Eq. (6), $\sin\theta_r = 0.5$. Since $-\pi/2 \le \theta_r \le \pi/2$, so that $\theta_r = 30$ deg. Then, by calculating Eqs. (4) and (5)

$$\cos\psi_r \cos\theta_r = 0.866 \tag{9a}$$

$$\sin\psi_r \cos\theta_r = 0 \tag{9b}$$

Since $\cos\theta_r > 0$, from Eq. (9a) $\cos\psi_r > 0$. Therefore, from Eq. (9b) $\psi_r = 0$. Next, calculating Eq. (8) and the preceding result $\theta_r = 30$ deg yields $\cos\phi_r = -1$, i.e., $\phi_r = 180$ deg. To sum up, $(\psi_r, \theta_r, \phi_r) = (0, 30, 180 \text{ deg})$. This example is illustrated in Fig. 2. When viewed from attitude 1, attitude 2 is also inverted and shows 30-deg pitch up.

**Example 2**

Attitude 1 is $(\psi_1, \theta_1, \phi_1) = (0, 0, 45 \text{ deg})$ and attitude 2 is $(\psi_2, \theta_2, \phi_2) = (30, 30, 45 \text{ deg})$. As shown in Fig. 3, it seems that attitude 2 is directly above when viewed from attitude 1, i.e., the relative attitude angle is only the relative pitch Euler angle $\theta_r$. But this is not strictly correct. According to the numerical

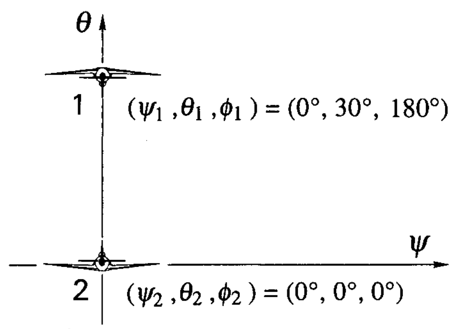


**Fig. 2 Same relative attitudes.**

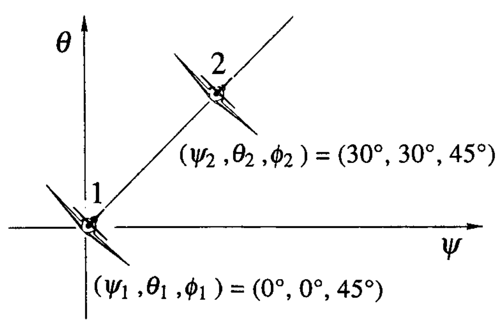


**Fig. 3 Relationship on the plane.**

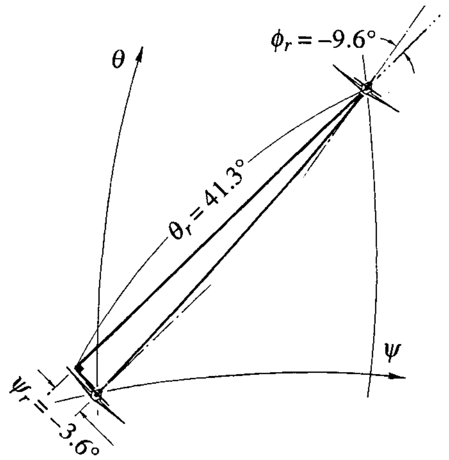


**Fig. 4 Exact relationship on the $\psi$-$\theta$ surface.**

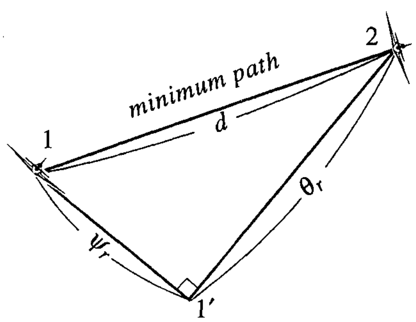


**Fig. 5 Relative Euler rotation triangle.**

calculations, $(\psi_r, \theta_r, \phi_r) = (-3.6, 41.3, -9.6 \text{ deg})$. Therefore, when viewed from attitude 1, attitude 2 may appear to be directly above but, in fact, is positioned slightly to the left and also banked slightly to the left. As seen from this, the relative attitude cannot be estimated by simply comparing both Euler angles. Figure 4 shows this exact relationship on the $\psi$-$\theta$ surface.

## Paths Between Two Flight Attitudes on the $\psi$-$\theta$ Surface

There are an infinite number of (attitude) paths between two flight attitudes on the $\psi$-$\theta$ surface. Here, only two special paths are considered (Fig. 5). One is the path defined by the relative Euler rotation sequence, i.e., the path 1-1′-2. Only when changing the attitude along this path by the relative Euler rotation sequence, the relative Euler angles $\phi_r$, $\theta_r$, and $\psi_r$ agree with the rolling, pitching, and yawing angles mentioned later, respectively. The second path is the minimum path.

### Minimum Path

As is well known, the (spherical) distance between two points on the surface of a sphere is defined as the length of the minor arc of the great circle that passes through the two points. The distance, i.e., the length of the minimum path from attitude 1 to attitude 2, can be defined on the $\psi$-$\theta$ surface in the same way. This can be calculated easily by making use of the relative Euler angles defined in the preceding section. If the relative Euler angles are obtained, a spherical, orthogonal triangle can be made of the sides $\psi_r$, $\theta_r$, and the minimum path (Fig. 5). Applying spherical trigonometry to this triangle, we get the distance $d$ as

$$\cos d = \cos \psi_r \cos \theta_r \tag{10}$$

which corresponds to the formula $d^2 = \psi_r^2 + \theta_r^2$ in plane trigonometry.

Note that the paths on the $\psi$-$\theta$ surface essentially do not include $\phi$. However, the $\phi$ can be estimated to some degree by the relationship between the direction of the path and of the lift. Also, note that the minimum path means the minimum trajectory drawn by the airplane's nose on the celestial sphere.

### Example 3

Attitude 1 is $(\psi_1, \theta_1, \phi_1) = (0, 70, 0 \text{ deg})$ and attitude 2 is $(\psi_2, \theta_2, \phi_2) = (150, 80, 180 \text{ deg})$. By the numerical calculations shown before,

$$(\psi_r, \theta_r, \phi_r) = (5.67, 28.6, 32.3 \text{ deg})$$

From Eq. (10) and this, one obtains

$$d = 29.1 \text{ deg}$$

Figure 6 illustrates this result.

## Rolling Angle, Pitching Angle, and Yawing Angle

The three Euler angles are the angles of a definite rotation sequence and attitude path with which an airplane can reach a flight attitude. Generally, however, they are not the rotation angles along the actual attitude path of a maneuver. Therefore, in this section, "rolling angle," "pitching angle," and "yawing angle" are defined as the rotation angles about the body-fixed axes along the actual attitude path. Unlike the Euler angles, these angles are airplane-centered angles. That is to say, for example, the phrase "pitching up" always means that the airplane's nose moves toward the body-fixed negative $Z$ axis regardless of the pitch Euler angle. Note that these terms are not general but are the terms suggested in this paper. The exact definitions are as follows.

Consider two flight attitudes 1 and 2, which are the flight attitudes of an airplane at $t = t_1$ and $t = t_2$, respectively. Then, the "rolling, pitching, and yawing angles" from the attitude 1 to attitude 2 are defined as integrated angular velocities

$$\text{rolling angle} = \int_{t_1}^{t_2} p \, dt \tag{11}$$

$$\text{pitching angle} = \int_{t_1}^{t_2} q \, dt \tag{12}$$

$$\text{yawing angle} = \int_{t_1}^{t_2} r \, dt \tag{13}$$

These angles show the rotation angles about the moving $X$, $Y$, and $Z$ axes, respectively, from $t = t_1$ to $t = t_2$.

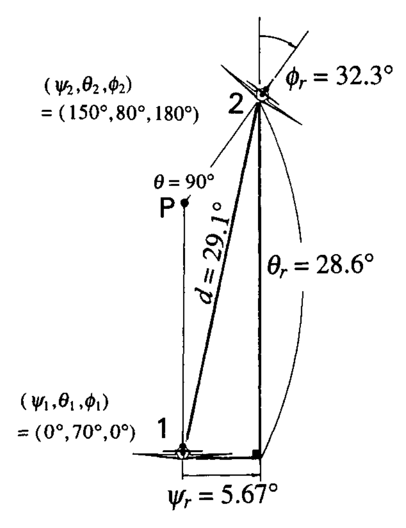


**Fig. 6 Distance between two attitudes.**

Also note that the relationships between the angular velocity components and the Euler angle derivatives are shown by the following kinematic equations:

$$p = \dot{\phi} - \dot{\psi} \sin \theta \tag{14a}$$

$$q = \dot{\theta} \cos \phi + \dot{\psi} \cos \theta \sin \phi \tag{14b}$$

$$r = \dot{\psi} \cos \theta \cos \phi - \dot{\theta} \sin \phi \tag{14c}$$

## Nose Moving Angle

According to the attitude projection method, a point on the $\psi$-$\theta$ surface represents an airplane's instantaneous flight attitude except for the roll Euler angle (bank angle). If necessary, the roll Euler angle can be added to the point by some graphical index as shown in the preceding figures. A trajectory on the $\psi$-$\theta$ surface ("$\psi$-$\theta$ trajectory," or "nose moving trajectory") represents the movement of this point with time. The velocity of (the point along) the $\psi$-$\theta$ trajectory, $w_{\psi\text{-}\theta}$ is shown by[2]

$$w_{\psi\text{-}\theta} = r\boldsymbol{e}_y - q\boldsymbol{e}_z \tag{15}$$

where $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ denote unit vectors of the $Y$ and $Z$ axes of an airplane, respectively. Also, $q$ and $r$ denote the pitching and yawing angular velocities, respectively. Note that the rolling angular velocity $p$ is not related to $w_{\psi\text{-}\theta}$, and a given vector $w_{\psi\text{-}\theta}$ does not determine the unique components $q$ and $r$. The $q$ and $r$ depend on the orientations of the basis vectors $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ with respect to the inertial space, i.e., they depend on the roll Euler angle at that moment.

The length of a nose moving trajectory defined on the $\psi$-$\theta$ surface is a kind of angle. The length of the trajectory $C$ from attitude 1 to attitude 2 equals the time integral of $|w_{\psi\text{-}\theta}|$. That is, it follows from Eq. (15) and Eqs. (14b) and (14c)

$$\int_{t_1}^{t_2} |w_{\psi\text{-}\theta}| \mathrm{d}t = \int_{t_1}^{t_2} \sqrt{q^2 + r^2}\, \mathrm{d}t$$

$$= \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 + \left(\cos\theta \frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2}\, \mathrm{d}t = \int_{1C}^{2} \mathrm{d}s \tag{16}$$

where $\mathrm{d}s$ is the line element. This angular quantity is called "nose moving angle" in this paper, since it is the angle that the airplane's nose moves. The nose moving angle is regarded as the composite angle of yawing angle and pitching angle. This angle can be used instead of considering the pitching angle and the yawing angle individually, because, as stated earlier, this angle is independent of the bank angle on the $\psi$-$\theta$ trajectory. The nose moving angle is the generalized angle which includes the "pitching angle," "yawing angle," and "turning angle" in special cases (Fig. 7). Geometrically, this is the length of the trajectory drawn by the airplane's nose movement on the $\psi$-$\theta$ surface. The "minimum path" defined before is the path along which the nose moving angle becomes smallest.

Although the "rolling angle" and the "nose moving angle" are both angles, they are somewhat different in meaning and not comparable with each other. For example, consider which is easier, 30-deg rolling or 30 deg turning. In conventional airplanes, the rolling angle is more quickly obtained than the turning angle. The rolling angle is almost immediately obtained by the aileron through the moment equations. Similarly, small amounts of the pitching angle and the yawing angle are obtained by the elevator and the rudder, respectively. But, these are chiefly due to the change or generation of the angle of attack $\alpha$ and the sideslip angle $\beta$ and are restricted within the magnitude limits of $\alpha$ and $\beta$. On the other hand, a large amount of the turning angle is not obtained until the flight-path angle changes through the force equations. Therefore, it is meaningless simply to compare both the magnitudes of the rolling angle and the nose moving angle.

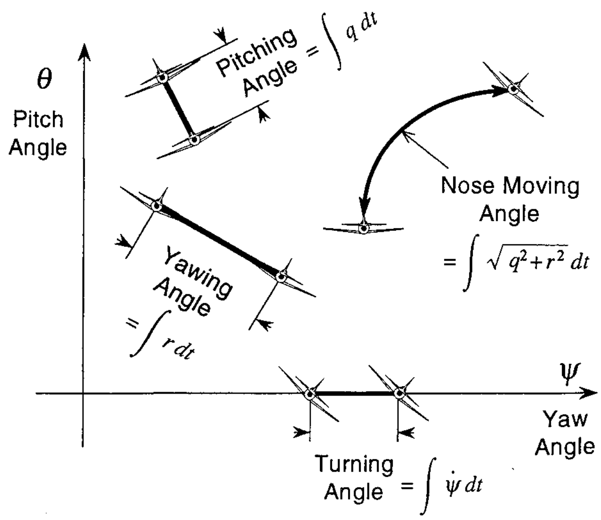


**Fig. 7 Several angles on the $\psi$-$\theta$ surface.**

## Rolling Angle and Roll Euler Angle

As mentioned previously, the rolling angle, pitching angle, and yawing angle are not the same as variations of roll Euler angle (bank angle), pitch Euler angle, and yaw Euler angle (heading angle), respectively. This is because the definitions differ. The difference between the "pitching angle" and the "variation of pitch Euler angle" is easy to understand. So is the difference between the "yawing angle" and the "variation of yaw Euler angle." Roughly speaking, the directions of their measurement axes are different. However, the difference between the "rolling angle" and the "variation of roll Euler angle" is not easily distinguished. Therefore, let us concentrate on the rolling angle in this section.

### Integration of $\dot{\psi} \sin\theta$

First, consider Eq. (14a). Integrating both sides of the equation from attitude 1 ($t = t_1$) to attitude 2 ($t = t_2$)

$$\int_{t_1}^{t_2} p\, \mathrm{d}t = \int_{t_1}^{t_2} \dot{\phi}\, \mathrm{d}t - \int_{t_1}^{t_2} \dot{\psi} \sin\theta\, \mathrm{d}t \tag{17}$$

The right-hand side of Eq. (17) can be rewritten

$$\int_{t_1}^{t_2} \dot{\phi}\, \mathrm{d}t - \int_{t_1}^{t_2} \dot{\psi} \sin\theta\, \mathrm{d}t = \int_{1}^{2} \mathrm{d}\phi - \int_{1}^{2} \sin\theta\, \mathrm{d}\psi \tag{18}$$

Therefore, Eq. (17) becomes

$$\int_{t_1}^{t_2} p\, \mathrm{d}t = \int_{1}^{2} \mathrm{d}\phi - \int_{1}^{2} \sin\theta\, \mathrm{d}\psi \tag{19}$$

The left-hand side is the rolling angle, and the first term on the right-hand side is the variation of roll Euler angle. Thus, the difference that is argued here depends on the second term on the right-hand side.

To integrate $\sin\theta$ with $\psi$, the integrand $\sin\theta$ must be given as a function of $\psi$, i.e., a path between attitude 1 and attitude 2 on the $\psi$-$\theta$ surface must be given. But since there are numerous possible paths from attitude 1 to attitude 2, the values of the integration itself are many. Accordingly the rolling angle similarly depends on the path. Therefore, it seems that we cannot examine it before a particular path is given, and that we cannot do so generally without the numerical calculation for each individual path. However, a general property of the rolling angle can be obtained from the following consideration on the $\psi$-$\theta$ surface.

### Application of Stokes Theorem

Figure 8 shows an arbitrary path from attitude 1 to attitude 2 on the $\psi$-$\theta$ surface. First, consider a closed path $C$: $1 \rightarrow 2 \rightarrow 2' \rightarrow 1' \rightarrow 1$, where the paths $1' \rightarrow 1$ and $2 \rightarrow 2'$ are the lines of $\psi$ = const, and pass through attitudes 1 and 2, respectively. Also, the path $2' \rightarrow 1'$ is the line of $\theta = 0$. Then, consider the following integral along this closed curve $C$.

$$\oint_C \sin\theta \, d\psi \tag{20}$$

By the additivity of path, this integral can be divided as follows:

$$\oint_C = \int_1^2 + \int_2^{2'} + \int_{2'}^{1'} + \int_{1'}^{1} \tag{21}$$

But $d\psi = 0$ on the paths $2 \rightarrow 2'$ and $1' \rightarrow 1$, and $\sin\theta = 0$ on the path $2' \rightarrow 1'$. Therefore, the integral (20) simply becomes

$$\oint_C \sin\theta \, d\psi = \int_1^2 \sin\theta \, d\psi \tag{22}$$

By Stokes theorem, the line integral on the left-hand side can be transformed into the surface integral (see Appendix):

$$\oint_C \sin\theta \, d\psi = \iint_S \cos\theta \, d\theta \, d\psi \tag{23}$$

where $S$ is the surface region enclosed by $C$, and the positive direction along $C$ is the clockwise direction viewed from the center of the $\psi$-$\theta$ sphere. Also, the sign of the surface area follows it. Now it can be seen that $\cos\theta \, d\theta \, d\psi$ on the right-hand side of Eq. (23) implies the surface element of a unit sphere $dS$: longitudinal line element $d\theta \times$ lateral line element $d\psi \cos\theta$. Thus the right-hand side of Eq. (23) becomes the area of $S$, i.e.,

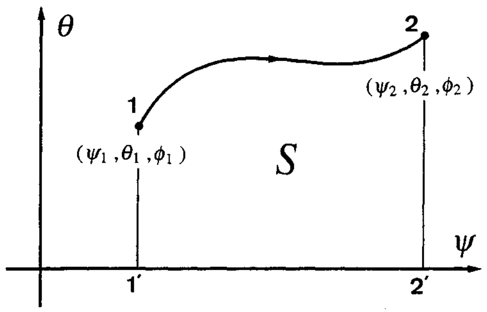


Fig. 8 Arbitrary path from attitude 1 to attitude 2.

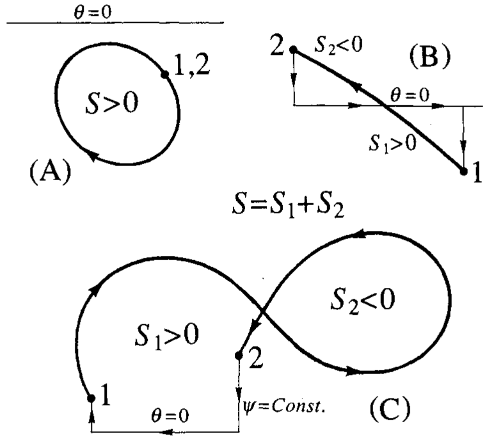


Fig. 9 Sign of area $S$.

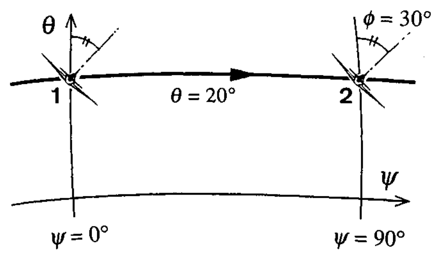


Fig. 10 Steady right high pitch-angle turn.

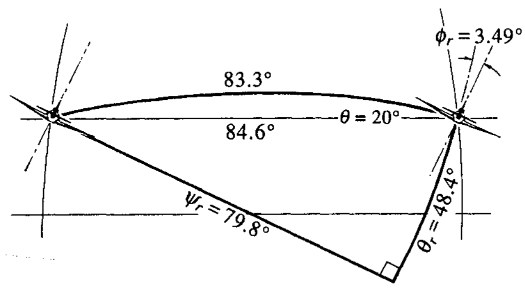


Fig. 11 Exact graphical relationship.

$$\iint_S \cos\theta \, d\theta \, d\psi = \iint_S dS \tag{24}$$

From Eqs. (22–24)

$$\int_1^2 \sin\theta \, d\psi = \iint_S dS \tag{25}$$

Finally, from Eqs. (19) and (25)

$$\int_{t_1}^{t_2} p \, dt = \int_1^2 d\phi - \iint_S dS \tag{26}$$

Accordingly, the difference between the variation of roll Euler angle and the rolling angle equals the surface area of $S$. Therefore, it can be said that the larger the area of $S$ becomes, the larger the difference becomes.

If the path is closed as in Fig. 9a, Stokes theorem can be applied directly ($C$ is the path itself). If the path is complicated or not closed, we can make more than one closed path, if necessary, by $\psi$ = const and $\theta = 0$ lines (Figs. 9b and 9c). In any case, whether the value of the surface area $S$ is positive or negative is determined by the direction of the path.

## General Examples

### High Pitch-Angle Turn

Consider a right turn where the pitch Euler angle is maintained at 20 deg, and the roll Euler angle (bank angle) is maintained at 30 deg. Figure 10 shows this turn. The turn at this pitch angle is usually a climbing turn except in high-angle-of-attack maneuvers because the flight-path pitch angle is usually also high. Attitude 1 and attitude 2 are the points of the yaw Euler angles (heading angles) $\psi = 0$ and $\psi = 90$ deg, respectively, in the turn. First, let us calculate the relative Euler angles of attitude 2 $[(\psi_2, \theta_2, \phi_2) = (90, 20, 30 \text{ deg})]$ relative to attitude 1 $[(\psi_1, \theta_1, \phi_1) = (0, 20, 30 \text{ deg})]$. By the numerical calculations from Eq. (4) to Eq. (8), one can obtain

$$(\psi_r, \theta_r, \phi_r) = (79.8, 48.4, 3.49 \text{ deg}) \tag{27}$$

Figure 11 shows this result graphically. Also the distance (the length of the minimum path) between attitude 1 and attitude 2 is obtained by Eqs. (10) and (27)

$$d = 83.3 \text{ deg}$$

Note that the original path (maintaining $\theta = 20$ deg) is not the minimum path. The nose moving angle along this path (or the length of the original path) is

$$90 \text{ deg} \times \cos 20 \text{ deg} = 84.6 \text{ deg}$$

Next, let us calculate the rolling angle during this 90-deg turn. In this case it is quite easy to calculate the rolling angle because the pitch Euler angle $\theta$ is constant. From Eq. (19) the rolling angle becomes

$$\begin{aligned}\int_{t_1}^{t_2} p \, dt &= \int_1^2 d\phi - \int_1^2 \sin\theta \, d\psi \\ &= \int_{30°}^{30°} d\phi - \int_{0°}^{90°} \sin 20° \, d\psi \\ &= (30° - 30°) - (\sin 20° \times 90°) \\ &= 0° - 31° = -31°\end{aligned}$$

That is, although the roll Euler angle (bank angle) is maintained, the airplane is rolling 31 deg to the left (rotates $-31$ deg about the airplane's $X$ axis). It may seem that this result is somewhat strange, but one can understand intuitively if one tries to turn a miniature aircraft by hand at an extremely high pitch Euler angle. In a right descending turn (to be precise, the right turn in a negative pitch Euler angle), the airplane is rolling to the right because the rolling angle becomes positive. If turning with the pitch Euler angle zero, the rolling angle is zero as long as the roll Euler angle does not change. In real flight, for example, the following is inferred from these results. If the pilot does not use the left aileron in the right, high pitch-angle turn previously stated, the right bank becomes deeper. Though the bank angle is maintained, the pilot must use the left aileron to generate left rolling moment. Therefore, it is harder for pilots to control the bank properly in high pitch angles than in usual pitch angles. The aileron is the control of the rolling rate $p$, not of the bank rate $\dot{\phi}$.

As seen from the value of the $\phi_r$ in Eq. (27), there is no direct relationship between the relative roll Euler angle and the rolling angle. In the assumption of small-disturbance, however, the "relative roll Euler angle," "rolling angle," and "variation of roll Euler angle" have almost the same values. Also note that only the rolling angle, among these angles, depends on the path between two flight attitudes.

### Large Flight Maneuver

Let us consider a large flight maneuver. Figure 12 shows the maneuver in which an airplane flies continuously pitching up and rolling to the right from attitude 1 $[(\psi_1, \theta_1, \phi_1) = (0, 0, 0 \text{ deg})]$ to attitude 2 $[(\psi_2, \theta_2, \phi_2) = (90, 0, 180 \text{ deg})]$, with its nose tracking a semicircle. This flight maneuver is like the upper-

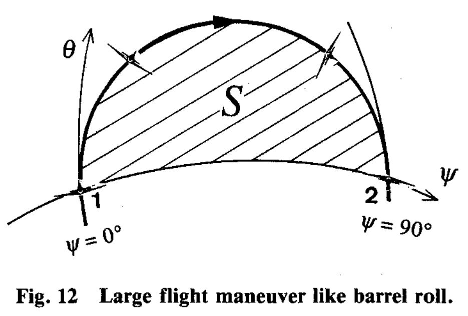


**Fig. 12 Large flight maneuver like barrel roll.**

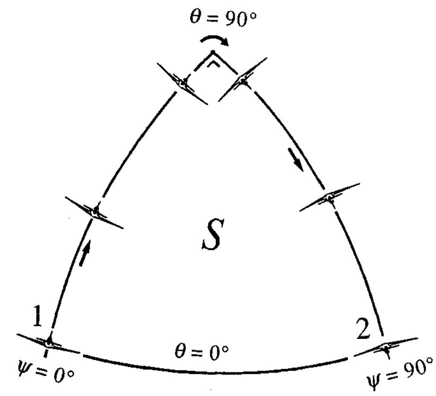


**Fig. 13 Extreme case.**

half of the barrel roll. It seems that the rolling angle from attitude 1 to attitude 2 along this trajectory is 180 deg because the roll Euler angle changes from 0 to 180 deg. But this also is not true. To calculate the rolling angle here, one may integrate $\sin\theta$ analytically along the semicircle. However, this is complicated. Therefore, we use Eq. (26).

The first term on the right-hand side of Eq. (26) is

$$\int_1^2 d\phi = \phi_2 - \phi_1 = 180 - 0 = 180 \text{ deg}$$

The second term is the area enclosed by both the semicircle and the line of $\theta = 0$ from attitude 1 to attitude 2 as shown in Fig. 12. The area of a circle on a unit sphere can be shown as

$$\pi[2 \sin(r/2)]^2$$

where $r$ is the radius of the circle (the spherical distance between the center of the circle on the spherical surface and each point on the circumference). In this case, since $r = 45$ deg $= \pi/4$ and $S$ is a half area of the circle, then

$$\iint_S dS = ½ \times \pi[2 \sin(\pi/8)]^2 = 0.92 = 52.7 \text{ deg}$$

Therefore,

$$\int_1^2 p \, dt = 180 - 52.7 = 127.3 \text{ deg}$$

That is, although the variation of roll Euler angle is 180 deg, the rolling angle is only 127.3 deg. Airplanes can be inverted with smaller rolling angles through high-pitch attitude paths.

On the other hand, the nose moving angle of this maneuver is the length of the semicircle from attitude 1 to attitude 2:

$$\int_{1C}^2 |w_{\psi\text{-}\theta}| \, dt = ½ \times 2\pi \times \sin 45 = 127.3 \text{ deg}$$

In this example, the numerical values of this angle and of the rolling angle happened to be the same.

### Extreme Case

This example is an extreme case of the preceding example and verifies its result intuitively. Figure 13 shows this flight maneuver. An airplane is pitching up from the starting point (attitude 1) to the point of 90-deg pitch Euler angle without rolling, and then is rolling 90 deg only at that point. Next, the airplane is pitching up (viewed from the cockpit of the airplane, but the pitch Euler angle decreases) to the point of 0-deg pitch Euler angle without rolling. Finally, at the ending point (attitude 2), the airplane becomes inverted, and the yaw Euler angle (heading angle) becomes 90 deg. The rolling angle

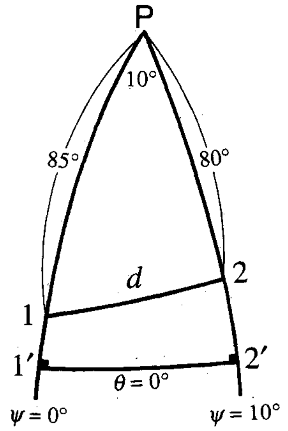


**Fig. 14 Small attitude change.**

of this flight maneuver from attitude 1 to attitude 2 is, as stated earlier, 90 deg. Also, the variation of roll Euler angle is 180 deg.

Now let us examine the rolling angle from the calculation. From Eq. (26)

$$\text{rolling angle} = \text{variation of roll Euler angle} - S$$

where $S$ is the area of the spherical triangle shown in Fig. 13, and 1/8 of the surface area of a unit sphere, i.e.,

$$S = (1/8)4\pi = \pi/2$$

Since the variation of roll Euler angle is 180 deg $= \pi$,

$$\text{rolling angle} = \pi - \pi/2 = \pi/2 = 90 \text{ deg}$$

This calculated value agrees with the initial description of this flight maneuver.

So far we have examined the differences between the "variation of roll Euler angle," "relative roll Euler angle," and "rolling angle" in large flight maneuvers. Finally, we check numerically how small those differences are in relatively small attitude changes.

#### Small Attitude Change

Attitude 1 is $(\psi_1, \theta_1, \phi_1) = (0, 5, 10 \text{ deg})$ and attitude 2 is $(\psi_2, \theta_2, \phi_2) = (10, 10, 30 \text{ deg})$. The path from attitude 1 to attitude 2 is assumed to be the minimum path (Fig. 14). First, by the numerical calculations as before, the relative Euler angles are

$$(\psi_r, \theta_r, \phi_r) = (8.865, 6.708, 19.21 \text{ deg})$$

and the length of the minimum path is

$$d = 11.10 \text{ deg} \tag{28}$$

The rolling angle is calculated by

$$\int_{t_1}^{t_2} p\, \mathrm{d}t = \int_1^2 \mathrm{d}\phi - \iint_S \mathrm{d}S$$

$$= (\phi_2 - \phi_1) - S_{\square 122'1'}$$

where $S_{\square 122'1'}$ denotes the area of the quadrangle $\square 122'1'$. Furthermore,

$$S_{\square 122'1'} = S_{\Delta 1'P2'} - S_{\Delta 1P2}$$

where $S_{\Delta 1'P2'}$ and $S_{\Delta 1P2}$ denote the areas of the triangles $\Delta 1'P2'$ and $\Delta 1P2$, respectively. Then, the former is

$$S_{\Delta 1'P2'} = ½ \times 4\pi \times [(\psi_2 - \psi_1)/2\pi]$$

And from spherical trigonometry, the latter is

$$S_{\Delta 1P2} = \angle 1P2 + \angle 21P + \angle P21 - \pi$$

The two unknown internal angles $\angle 21P$ and $\angle P21$ of the triangle $\Delta 1P2$ are obtained by the law of sines in spherical trigonometry and Eq. (28). By numerical calculations, we get finally the rolling angle, 0.3262 = 18.69 deg. To sum up

$$\text{variation of roll Euler angle} = \int \mathrm{d}\phi = 20 \text{ deg}$$

$$\text{relative roll Euler angle} = \phi_r = 19.21 \text{ deg}$$

$$\text{rolling angle} = \int p\, \mathrm{d}t = 18.69 \text{ deg}$$

In addition, if attitude 1 is $(\psi_1, \theta_1, \phi_1) = (0, 0.5, 10 \text{ deg})$ and attitude 2 is $(\psi_2, \theta_2, \phi_2) = (1, 1, 30 \text{ deg})$, then by similar calculations we can obtain

$$\text{variation of roll Euler angle} = 20 \text{ deg}$$

$$\text{relative roll Euler angle} = 19.992 \text{ deg}$$

$$\text{rolling angle} = 19.987 \text{ deg}$$

and the differences become much smaller.

## Concluding Remarks

A reconsideration of angles used to describe airplane flight maneuvers has been made. The "relative Euler angles," "rolling angle," "pitching angle," "yawing angle," and "nose moving angle" were proposed and defined in a theoretical and practical way. Calculations and comparisons of similar angles in the small-disturbance assumption were then made for both small and large flight maneuvers. It was made clear that although the "variation of roll Euler angle," "relative roll Euler angle," and "rolling angle" can be considered approximately equal in the small-disturbance theory, they are obviously different from each other in large flight maneuvers. As an interesting result, a general property and calculation method for the rolling angle was found. The new angles presented in this paper can be useful as basic quantities for analyzing large flight maneuvers.

## Appendix

The Stokes theorem can be shown most simply by using differential forms as follows:

$$\int_{\partial S} \omega = \iint_S \mathrm{d}\omega \tag{A1}$$

where $\omega$ is a differential form, and $\mathrm{d}\omega$ is its exterior differential (for further details, see mathematics book on differential forms). Now, if $\omega$ is the differential form given by

$$\omega = \sin\theta\, \mathrm{d}\psi \tag{A2}$$

Then, its exterior differential $\mathrm{d}\omega$ is

$$\mathrm{d}\omega = \cos\theta\, \mathrm{d}\theta \wedge \mathrm{d}\psi \tag{A3}$$

where $\wedge$ denotes the exterior product. Therefore, Eq. (A1) becomes

$$\int_{\partial S} \sin\theta\, \mathrm{d}\psi = \iint_S \cos\theta\, \mathrm{d}\theta\, \mathrm{d}\psi \tag{A4}$$

## References

[1]Kalviste, J., "Spherical Mapping and Analysis of Aircraft Angles for Maneuvering Flight," *Proceedings of the AIAA Atmospheric Flight Mechanics Conference,* AIAA, New York, 1986, pp. 486–495.

[2]Kato, O., "Attitude Projection Method for Analyzing Large-Amplitude Airplane Maneuvers," *Journal of Guidance, Control, and Dynamics,* Vol. 13, No. 1, 1990, pp. 22–29.

[3]Kato, O., and Sugiura, I., "An Interpretation of Airplane General Motion and Control as Inverse Problem," *Journal of Guidance, Control, and Dynamics,* Vol. 9, No. 2, 1986, pp. 198–204.